Kinematics

Introduction
Motion is fundamental to our lives and to our thinking. Moving from place to place in a given amount of time helps define both who we are and how we see the world. Seeing other people, objects or animals moving and being able to imagine where they came from, where they're going and how long it'll take them to get there is natural to us.

All animals, not just humans, do calculations about the motion of themselves and the world around them. Without that ability we could not survive. A basic understanding of motion is deep in our minds and was there long before we could write or talk about physics.

It should be no surprise that the first, and most fundamental, questions of physics relate to motion. Many of the first writings of physics are on this topic and date back thousands of years. The study of motion is called kinematics. It comes from the Greek words kinema, which means motion. Almost everything we learn in physics will involve the motion of objects. So, kinematics must be understood well in order to understand other topics we will be studying in the future.

Time and Distance
Everyone knows what time and distance are until they're asked to define them. Go ahead; try to define what time is without using the idea of time itself in your definition. Here are some definitions we've heard before. I'm sure you'll come up with some new ones as well.

Time is the amount of time that goes by.
Time is how long it takes for something to happen.
Time is how long I have to wait.

The problem with these definitions is that they use the word "time" in the definition, or imply its use. In the first definition, if you don't know what time is, how can you use it to define time? In the second two definitions, the phrase "how long" is just another way of saying "the amount of time". They don't qualify as definitions since if you didn't know what time was in the beginning; you still don't.

See if you can define distance. We think you'll run into the same problem.

We all believe that we know what these terms mean, but it's impossible to define them. We move through time and space as naturally as a fish moves in the water. Time and space are all around us, but we can't really say what they are. They are too fundamental to be defined. We have to take them as "givens". You and I have a sense that we know what they are. That sense comes directly from our minds and bodies; but we can't really define them beyond that.

Our sense of time and distance must have evolved in us long before we could think about them. All animals need a basic perception about time and distance in order to survive. The most primitive one celled animals move about through time and space and surely they don't have a definition of what those concepts represent. The sense of time and distance predates our ability to think and perhaps that's why we can't use our minds to define them; but we can work with them.

We can **measure** the flow of time with clocks and the distance using a ruler. These don't represent definitions; but they do allow us to compare different intervals of time and space with one another. The ability to measure time and distance represents a starting point for physics.

For instance, let's say the amount of time it takes for me to run once around a track is 2 minutes. That means that the minute hand of my stop watch will go around two times while I run around the track one time. That doesn't tell me what time is, but it does tell me that those two processes took the same amount of it. So I can compare the time it takes for something to happen to the time it takes for something else to happen.

Similarly, I don't need to know what distance is in order to compare the distance between two objects with the distance between two other objects. I can say that the distance from the heel to the toe of my foot is the same as the distance from one end to the other of a one foot ruler. So I can say that my foot is one foot long, even without a definition of what length means.

Units of Time and Distance
In order that people can compare their measurements with those taken by others; an international system of measurements was agreed upon. The System International (SI) is used by virtually all scientists in the world. In that system, the basic unit of length is the meter and the basic unit of time is the second.

Distance is measured in meters (m).
Time is measured in seconds (s).

Length measurements are made by comparing the distance between any two locations to the distance between the two ends of a rod that is defined to be one meter long. Time measurements are made by measuring the time between events with the time it takes for a second to tick off on a clock

Scientists use the meter, instead of the foot, for measuring distances because it's simpler. Using the SI system to measure lengths in meters, is no more accurate than using the English system measuring lengths in feet. However, it turns out to be a lot easier. That's because the mathematics of dealing with 12 inches in a foot and 5280 feet in a mile is just a lot more difficult than the metric system of 100 centimeters in a meter and 1000 meters in a kilometer. When you start solving problems, you'll be happy not having to deal with feet, miles and inches.

Constant Speed

If everything in the world just stood still, we'd need to measure time and distance separately and we'd be done. But that'd be a pretty boring world. Many of the most interesting things involve motion; objects moving from one location to another in a certain amount of time. How fast they do this is the **speed** of the object.

Speed is not a fundamental property of the world, like distance and time, but is a human invention. It is defined as the ratio of the distance traveled divided by the time it took to travel that distance.

$$\mathbf{Speed} \equiv \frac{\mathbf{Distance}}{\mathbf{Time}}$$

or

$\boldsymbol{s} \equiv \frac{\mathbf{d}}{\mathbf{t}}$

The equal sign with three parallel lines just points out that this is a definition. We made up the word "speed" and then defined it to mean the ratio of distance and time. There's no way that this could be proven right or wrong, through experiment or any other means, since we just made it up. When we use the formula, we'll usually just write it with a normal equals sign, but we should remember that it's just our definition.

Speed and distance don't depend on the direction traveled. So if you walk two miles to school and then return back home, the total distance you traveled is four miles. If it took you one hour to do that, your average speed was four miles per hour. In this section we're assuming that your speed is constant. In later sections, we'll talk about cases where your speed changes.

Units of Speed
The units of speed can be derived from its formula:
$s = \frac{d}{t}$
The SI unit of distance is meter (m) and time in second (s). Therefore, the unit of speed is m/s.

Problem Solving

When solving physics problems there is a series of steps that should be followed. In the early problems that we'll be doing, it'll be possible to skip some steps and still get the correct answer. But that won't give you a chance to practice the methods that you'll need to solve more difficult problems. It's wise to learn how to swim in the shallow end of the pool, but if all you do is stand up there, it won't be much help when the water gets deep. So, please use the following approach on all the problems you solve right from the beginning. It'll pay off in the end.

1. Read the problem carefully and underline, or make note of, any information that seems like it may be useful.
2. Read the problem through again, but now start writing down the information that will be of value to you. Identify what is being asked for and what is being given.
3. If appropriate, draw a sketch.
4. Identify a formula that relates to the information that you've been given to the information you've been to asked to solve for.
5. Rearrange the formula so it's solved for the variable you're looking for. This means, get that variable to be alone on the left side of the equals sign.
6. Substitute in the values you've been given, including units.
7. Calculate the numerical result.
8. Solve for the units on the right side of the equation and compare those to the units that are appropriate for what you're solving for. For instance, if you're solving for distance, the units should be in meters not meters per second.
9. Reread the problem and make sure that your answer makes sense. It's been shown that successful physics students read each problem at least three times.

Example 1: Riding your bike at a constant speed takes you 25 seconds to travel a distance of 1500 meters. What was your speed?

We've been given distance and time and we need to find speed.

s = ?
d = 1500 m
t = 25 s

We can directly use the equation $s = \frac{d}{t}$. That gives the relationship between the three variables and is already solved for the variable that we're looking for. After writing down the formula we just have to substitute in the values with units.

$s = \frac{d}{t}$

$s = \frac{1500m}{25s}$

s = 60 m/s

Note that not only is 1500 divided by 25 equal to 60 but also that m divided by s yields m/s which is the correct unit for speed. By always doing the same mathematical operations on the units as well as the numbers you should end up with the correct units in your answer. This is a good way to check if you did the problem correctly.

Let's now looks at an example where the formula can't be used directly.

Example 2: How far will you travel if you are driving at a constant speed of 25 m/s for a time of 360s?

We've been given speed and time and we need to find distance.

s = 25m/s
d = ?
t = 360s

We'll use the same formula (s=d/t) since it relates the known values (speed and time) to the unknown value (distance). But, in this case while we need to solve for distance (d), the formula that we have ($s = \frac{d}{t}$) is solved for speed (s). We must first use algebra to rearrange the formula so that we solve for d. Once you reach this point, you substitute the values into the formula.

We'll make use of three rules to rearrange the formula.

1. If the variable that we are solving for is in the numerator and isn't alone, then it is mathematically connected to other numbers and/or variables. We can get it alone by performing the inverse operation on each of the other variables or numbers. For instance, if d is divided by t; we can get d alone by multiplying by t (since multiplication is the opposite of division).
2. We can do anything we want to one side of an equation as long as we also do it to the other side (except dividing by zero). So if we multiply the right side of the equation, d/t, by t we also have to multiply the left side of the equation, s, by t.

3. We can always switch the right and left sides of an equation.

Let's use this approach to solve the equation, $s = \frac{d}{t}$, for d.

$s = \frac{d}{t}$

Since we are solving for d, and d is being divided by t, we must multiply d/t by t. But we can only do that if we multiply s by t. So multiply both sides by t.

st = $(\frac{d}{t})$t

Cancel t on the right since it's in the numerator and the denominator and t/t =1

st = d

Switch the d to the left side

d = st

Substitute in values for s and t

d = $(25\frac{m}{s})$ (360s)

d = 9000m

Note that not only is 25 times 360 equal to 9000 but also that meters per second times seconds is equal to meters since the seconds will cancel out. That gives us units of meters, which makes sense since we are solving for a distance.

Example 3: How much time will it take you to travel 3600 m if you are driving at a constant speed of 20 m/s?

We've been given the distance and the speed, and we need to find the time.

s = 20m/s
d = 3600m
t = ?

In this case we are solving for time (t) but the formula we have (s = d/t) is solved for speed (s). We must first use algebra to rearrange the formula so that it is solved for t. Only at that point should values be substituted into the formula s = d/t. We need to add one additional rule to rearrange the formula in this case.

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4. The unknown for which we are solving must be in the numerator, not the denominator. So if we are solving the formula s = d/t for t, our first step must be to move t to the numerator on the left instead of leaving it in the denominator on the right. To do this, we need to multiply both sides of the equation by t, giving us st = d. Then we can proceed just as we did above.

$s = \frac{d}{t}$

Multiply both sides by t to cancel the t on the right and get it on the left

$st = \left(\frac{d}{t}\right) t$

Cancel t on the right since t/t = 1

$st = d$

Since t is not alone, because it is multiplied by s, we must divide both sides by s

$\frac{st}{s} = \frac{d}{s}$

Cancel s on the left side since s/s = 1

$t = \frac{d}{s}$

Substitute in values for d and s

$t = \frac{3600m}{20\frac{m}{s}}$

$t = 180s$

The units may be a bit harder to understand in this case. We have meters divided by meters per second. But you may recall from fractions that dividing by a fraction is the same as multiplying by its reciprocal. (Dividing by 1/3 is the same as multiplying by 3.) So dividing by m/s is the same as multiplying by s/m. This makes it clear that the meters will cancel out, when we multiply s/m by m, and we are left with seconds, an appropriate unit for time.

<u>Instantaneous Speed</u>

There's an old joke about a person who's pulled over for speeding. The police officer tells the speeder that he was going 60 miles per hour in a forty mile per hour zone. The speeder's response is that he couldn't have been going sixty miles per hour since he'd only been driving for fifteen minutes.

The reason that argument doesn't work is that speed is a ratio of distance and time. There are an infinite number of ways that you can calculate a speed of ten meters per second. Some are shown in the table below.

Distance (m)	Time (s)	Speed (m/s)
1000	100	10
500	50	10
100	10	10
10	1	10
1	0.1	10
0.1	0.01	10
0.01	0.001	10
0.001	0.0001	10

You can see that at the bottom of the chart that if you travel one thousandth of a meter in one ten thousandth of a second you are traveling at a speed of ten meters per second. That time and distance can be made as small as you like. When the time over which the speed is measured is very small, the speed that is calculated is called the **instantaneous speed**. This is the speed that you read on your speedometer or that a policeman reads on his radar or laser gun.

Average Speed

While traveling along, your varies; to go up and down along the way. You might even stop for a while to have lunch. Your instantaneous speed at some moment during your trip and your average speed for the total trip are often not the same. Your average speed is calculated by determining the total distance that you traveled and dividing by the total time that it took you to travel that distance.

Example 4:
You ride your bike home from school by way of your friend's house. It takes you 7 minutes to travel the 2500m to your friend's house. You then spend 10 minutes there. You then travel the 3500 m to your house in 9 minutes. What was your average speed for your total trip home?

Your average speed will be obtained by dividing the total distance traveled by the total time it took to travel that distance. In this case, the trip consisted of three segments. The first segment (I) is the ride to your friend's house, the second segment (II) was the time at your friend's house and the third segment (III) was your ride home from your friend's house. In the chart below, the speed is calculated for each segment, even though that is not necessary to get the answer that was requested, the average speed for your total trip. All calculated figures are shown in bold type.

Segment	Distance (m)	Time (s)	Speed (m/s)
I	2500	420	**6.0**
II	0	600	**0.0**
III	3500	540	**6.5**
Total/ Average	**6000**	**1560**	**3.8**

For instance the speed for the first segment is given by:

s = ?
d = 2500m
t = 7 minutes = 420 seconds

Note that we need to convert the given time in seconds in order to use SI units. Since there are 60 seconds in a minute, this requires multiplying seven minutes by the fraction (60 seconds / 1 minute). This leaves us with 420 seconds.

$s = \frac{d}{t}$

$s = \frac{2500m}{420s}$

s = 6.0 m/s

For the second segment, your speed was zero since you were within the house. But even though you weren't moving, time was going by. So the 10 minutes, or 600 seconds, still counts towards the total elapsed time.

The third segment is calculated in the same manner as was the first.

s = ?
d = 3500m
t = 9 minutes = 540 seconds
$s = \frac{d}{t}$
$s = \frac{3500\text{m}}{540\text{s}}$
s = 6.5 m/s

The average speed is calculated by taking the total distance, 6000m, and dividing it by the total time, 1560s, to get an average speed of 3.8 m/s.

While it wasn't necessary in this case to calculate the speed for each interval, it's important to note that the average speed is not the average of the speeds. The average of 6.0 m/s, 0.0 m/s and 6.5 m/s is 4.2 m/s. But this is not the correct answer. The correct answer can only be obtained by first finding the total distance and dividing that by the total time, by doing this you get the answer of 3.8 m/s.

Example 5:
You run a distance of 210 m at a speed of 7 m/s. You then jog a distance of 200 m in a time of 40s. Finally, you run for 25s at a speed of 6 m/s. What was the average speed of your total run?

Your average speed will be obtained by dividing the total distance traveled by the total time it took to travel that distance. In this case, the trip consisted of three segments. In the chart below, different calculations are required for each segment in order to obtain the average speed for your total trip.

Segment	Distance (m)	Time (s)	Speed (m/s)
I	210	**30**	7.0
II	200	40	**5.0**
III	**150**	25	6.0
Total / Average	**560**	**95**	**5.9**

The time for the first segment is given by:

s = 7.0 m/s
d = 210m
t = ?

s= $\frac{d}{t}$
st = d
t = $\frac{d}{s}$
t = $\frac{210m}{7\frac{m}{s}}$
t = 30 s

We don't really need to calculate your speed for the second segment, but we'll do it anyway.
s = ?
d = 200m
t = 40s

s = $\frac{d}{t}$
s = $\frac{200m}{40s}$
s= 5.0 m/s

The distance needs to be calculated for the third segment.
s = 6.0 m/s
d = ?
t = 25s

s = $\frac{d}{t}$
st = d
d = st
d = (6.0 m/s) (25s)
d = 150 m

The average speed is calculated by taking the total distance, 560m, and dividing it by the total time, 95s, to get 5.9 m/s.

Position, Displacement and Velocity
So far our analysis has not required, or even allowed, us to know anything about the direction of the motion under study. But in real life, direction is usually very important. Whether you're driving 60 miles per hour north or 60 miles per hour south, it makes a great deal of difference as to where you end up.

Scalars are quantities that are defined only by their **magnitude**; the numerical value. Speed, time and distance are all examples of scalars. When we speak of 40 m/s, 20 minutes or 3 miles we're not giving any information about direction.

Vectors are quantities that are defined by both the **magnitude** and **direction**. So, instead of saying that I traveled a distance of 400m, I would say that I traveled 400m north; I am now defining vector. The vector that is or defined by combining distance with direction is called **displacement**. The symbol for displacement is "**Δx**". We'll talk more about that symbol a little later, but you can use it in the meantime. Also, in order to keep track of what's a scalar and what's a vector, we'll always show vectors in a bold typeface.

There are important differences when we work with scalars and vectors. This differences can be most easily seen by using distance and displacement as examples. For instance, while distance are always positive, since they have no direction associated with them, displacement can be positive or negative. That means that if I were to take a trip which involved going 200m north and then 200m south I get very different answers for the total distance I traveled and my total displacement. I get my total distance by adding 200m to 200m and getting 400m. That's the total distance that I walked.

On the other hand, my displacement represents the sum of the two displacements. My initial displacement north is equal and opposite to my final displacement south, so they will cancel each other out. If I think of north as the positive direction, the first displacement would be +200m, while my second displacement would be -200m, since it's treveling to the south. The sum of +200m and -200m is zero. That's because the direction of the motion matters with displacement while it doesn't apply to distance. As a result, displacement tells you how far you are from where you started. In this case, I am zero distance from where I started, since I end up back where I started off.

Example 6:
You drive 1500m north and then 500m south. Determine both the total distance you traveled and your total displacement from where you started.

The distance traveled is just the sum of the two distances, 1500m and 500m, 2000m.

In order to determine your total displacement we need to first define our directions. Let's call movement to the north positive and movement to the south negative (which direction we call positive won't affect our answer as long as we're consistent.). That means that for the first part of the trip your displacement is +1500m and for the second part of the trip your displacement is -500m. Your total displacement is the sum of those two, +1000m. Since we decided that we'd call the north direction positive, your final displacement is 1000m north. The last step of converting from +1000m to 1000m north is important in that our choice of + or – was arbitrary so we need to translate back to the original directions we were given in the problem.

The important point here is that the answers are different and have different uses. The distance you traveled, 2000m, tells you something about how tired you may be because it tells you the total distance you had to move yourself during this trip. Your displacement, 1000m north, tells you where you are at this point in your travels relative to where you started.

The same difference exists between speed and **velocity**. The symbol for velocity is **v** and the symbol for average velocity is $\mathbf{v_{avg}}$. The average velocity is determined by dividing your total displacement by the time it took for that displacement. This is similar to how we calculated average speed by dividing the total distance traveled by the total time it took to travel that distance.

$s \equiv \frac{d}{t}$ while $v_{avg} \equiv \frac{\Delta x}{t}$

Example 7:
If the travel in Example 6 was done at constant speed and required a total time of 500s, determine the average speed and the average velocity.

s = ?
d = 2000m
t = 500s

$s = \frac{d}{t}$

$s = \frac{2000m}{500s}$

s = 4.0 m/s

v_{avg} = ?
Δx = 1000m north
t = 500s

$v_{avg} = \frac{\Delta x}{t}$

$v = \frac{1000m\ North}{500s}$

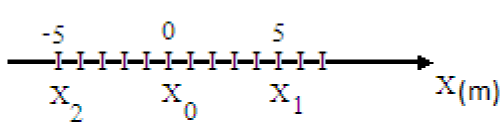
v = 2m/s North

Note that the numerical answers are different and that the answer for velocity includes a direction while the answer for speed does not.

Coordinate Systems

The displacement of an object tells us how its **position** has changed. In order to better understand what that means we need a way of defining position; we need a **coordinate system**.

The requirements of any coordinate system are an origin and an orientation. In other words, you need to pick a zero from which you'll be making measurements and you need to know the direction in which you will be measuring. The simplest type of coordinate system is one-dimensional, in which case the coordinate system becomes a number line, as shown below.



The origin is located at zero, negative positions are to the left of the origin and positive positions are to the right. We can identify different locations on the number line as such as x_0, x_1 and x_2. In the diagram above, x_0 is located at 0, x_1 is located at +5m and x_2 is located at -5m.

We can now refine our definition of displacement, the change in the position of an object, as being the difference between an object's final position, x, and its initial position, x_o. It now becomes clear why the symbol for displacement is "Δx". The Greek letter delta, "Δ", means "the change in" so "Δx"

can be read as "delta x" or "the change in x". Symbolically this becomes;
$\Delta x \equiv x - x_o$

Example 8:
An object moves from an initial position of +5m to a final position of +10m in a time of 10s. What displacement did it undergo? What was its average velocity?

$x = +10m$
$x_o = + 5m$
$\Delta x = ?$

$\Delta x = x - x_o$
$\Delta x = (+10m) – (+5m)$
$\Delta x = +5m$

$v_{avg} = ?$
$\Delta x = +5m$
$t = 10\ s$

$v_{avg} = \frac{\Delta x}{t}$
$v = \frac{+5m}{10s}$
$v = +0.5m/s$

Example 9:
An object moves from an initial position of +5m to a final position of -10m in a time of 0.25s. What displacement did it undergo? What was its average velocity?

$x = -10m$
$x_o = + 5m$
$\Delta x = ?$

$\Delta x = x - x_o$
$\Delta x = (-10m) – (+5m)$
$\Delta x = -15m$

$v_{avg} = ?$
$\Delta x = -15m$
$t = 0.25\ s$
$v_{avg} = \frac{\Delta x}{t}$
$v_{avg} = \frac{-5m}{.25s}$
$v_{avg} = -60m/s$

Once again, notice that the answer includes a **magnitude**, 15m, as well as a **direction**, "-".

Vectors, such as displacement or velocity, can be depicted as an arrow. The length of the arrow represents the magnitude of the vector and the direction it is pointed represents the direction of the vector.

Vectors can be added either graphically or algebraically. (Even if you're solving a problem algebraically it's helpful to also sketch the addition graphically so you can make certain that your answer makes sense.) The way to add vectors graphically is to draw the first vector starting at the origin of the problem. It must be drawn to scale and pointed in the correct direction. The second vector should be drawn in the same manner, but starting where the first one ended. The sum of the two vectors is simply a third vector which starts where the first vector started and ends where the last vector ended. In other words, the solution is a third vector that connects the beginning of the first to the end of the last vector drawn. The arrow tip of that vector should point away from the location from which it started.

The following example is solved both graphically and algebraically.

Example 10:
Beginning at a location that's 400 m east of your home, you travel 500m east and then 300m west. How far are you now from your home? What displacement have you experienced during your trip? If this travel took a total time of 20s, what was your average velocity?
The graphical solution, shown below, starts by drawing a sufficiently large east-west axis. If your home is located at x = 0m, then your initial position, x_o, is 400m east of that. Draw a vector that describes the first part of your trip by drawing an arrow that begins at the location 400m to the east of your house, is 500m long and points towards the east. That tip of that arrow should then be 900m east of your house. Then draw the vector for the second part of your trip by drawing an arrow that begins at the tip of the first arrow, 900m east of your house, is 300m long and is pointed to the west, towards your house. The tip of that arrow should now lie at a location 600m to the east of your house; that is your final location, x.

Your displacement is the difference between your final and initial positions. This is obtained, graphically, by drawing an arrow that starts at your initial position and ends at your final position. The length of this arrow, which can be physically measured or read off the scale, is the magnitude of your displacement. The direction of the arrow is the direction of your displacement. As is shown below, it can be seen that your displacement is 200m east.

Your average velocity is your total displacement divided by the total time it took to undergo that displacement. In this case, we graphically determined that your displacement is 200m east and we were told that your travel time was 20s. So,

v_{avg} = ?
Δx = 200m east
t = 20s

$v_{avg} = \frac{\Delta x}{t}$
$v_{avg} = \frac{200m\ east}{20s}$
v_{avg} = 10m/s east

The same problem can be solved algebraically, although an initial sketch is still a good idea. The key step to an algebraic solution is to convert directions to be either positive or negative. In this case, we can define east as positive and west as negative (The choice won't matter as long as we're consistent throughout the problem.)

Your initial position then becomes +400m (400m east), your initial travel is +500m (500m east) and the last leg of your trip is -300m (300m west). You can now just add these together to get your final position, +600m. This translates into a final position of 600m east of your house.

Your displacement is just the change in your position.

x = +600m
x_o = +400m
Δx = ?

Δx = x - x_o
Δx = +600m – (+400m)
Δx = +200m
Δx = 200m east   note that this last step is required since our choice of positive or negative was arbitrary

The calculation of average velocity can be done just as it was above for the graphical solution.

Instantaneous Velocity and Acceleration

The most boring world would be one in which the positions of all objects were constant...nothing would move: velocity would have no meaning. Fortunately our world is a lot more interesting than that. Objects are changing their positions all the time, so velocity is an important concept.

But our world is even more interesting, objects are also changing their velocity all the time: they are speeding up, changing direction and/or slowing down. Just as change in position over time leads to the idea of velocity, change in velocity over time leads to the concept of acceleration.

In the same manner that we defined instantaneous speed as the speed measured during a very short period of time, we can now define **instantaneous velocity** as the velocity measured during a very short period of time. The symbol, "**v**" will be used for instantaneous velocity. In a world with acceleration, the idea of instantaneous velocity is very important since an object's velocity may often be changing from moment to moment.

$v \equiv \frac{\Delta x}{t}$ for a very short period of time...an instant

We can now define acceleration as the change in velocity over time.

$a \equiv \frac{\Delta v}{t}$
or
$a \equiv \frac{v - v_0}{t}$

Units of Acceleration
The units of acceleration can be derived from its formula:
a ≡ $\frac{\Delta v}{t}$
The SI unit of velocity is meters/second (m/s) and of time is the second (s). Therefore, the unit of acceleration is (m/s)/s or m/s/s.
This is the same as (m/s) x (1/s) since dividing by s is the same as multiplying by 1/s.
This results in m/s^2 which, while not having any intuitive meaning, is a lot easier to keep track of than m/s/s, meters per second per second, the alternative way of writing the units for acceleration.

Since velocity is a vector, it has a magnitude and a direction. For the rest of this chapter, we'll be focused on accelerations that change the magnitude of an object's velocity. However, in later chapters a key aspect of acceleration will involve changing the direction of an object's velocity. These are both examples of acceleration. But let's first start with accelerations that change only the magnitude of an object's velocity.

Example 11
An object is traveling at a velocity of 20m/s north when it experiences an acceleration over 12s that increases its velocity to 40m/s in the same direction. What was the magnitude and direction of the acceleration?

Let's solve this algebraically by defining velocities towards the north as positive and towards the south as negative. Then,
v = +40m/s
v_0 = +20m/s
t= 12s
a = ?

a ≡ $\frac{\Delta v}{t}$
a ≡ $\frac{v-v_0}{t}$
a = $\frac{(+40\frac{m}{s})-(+20\frac{m}{s})}{12s}$
a = $\frac{+20\frac{m}{s}}{12s}$
a = +1.7m/s^2

Example 12
What will an object's velocity be at the end of 8.0s if its initial velocity is +35m/s and it is subject to an acceleration of -2.5m/s^2?

v = ?
v_0 = +35m/s
t= 8.0s
a = -2.5m/s^2

$a \equiv \frac{\Delta v}{t}$

$a \equiv \frac{v - v_0}{t}$

Solve for v: First, multiply both sides by t

$at = v - v_o$

Then add v_o to both sides

$v_o + at = v$

Switch so that v is on the left side of the =

$v = v_o + at$

Substitute in values and solve

$v = +35m/s + (-2.5m/s^2)(8.0s)$
$v = +35m/s + (-20m/s)$
$v = +15m/s$

Free Fall

You now know enough to be able to understand one of the great debates that marked the beginning of what we now call physics. The term "physics" was being used by the ancient Greeks more than 2000 years ago. Their philosophy, much of it described in the book titled "Physics" by Aristotle, included some ideas that stood until Galileo made some important arguments and measurements that showed the ancient Greek physics to be of limited value.

The physics of ancient Greece included the idea that all objects were made up of a combination of four elements (the fifth element was reserved for objects that were beyond the earth). The four elements to be found in our world were earth, water, air and fire. Each of these elements had their natural place. If you removed an element from its natural place, it would, when released, immediately move back to that place; and it would do so with its natural (constant) velocity.

Their view of the world could be thought of a set of concentric circles with each of the elements occupying a layer. Earth occupied the center of the circle, so rocks, which are predominantly made of earth would naturally move down, towards the center of our world. Above earth was water, which would fill the area above the rocks, like a lake or an ocean above the land that forms the lake or ocean bed. Above water is air, which is seen everywhere in our world, above both earth and water. Finally, fire rises up through the air, searching for its natural location above everything else.

All objects were considered to be a mixture of these four elements. Rocks were predominantly earth: so if you drop a rock it falls as it tries to get back to its natural location at the center of the earth. In so doing, it will pass through water and air: If you drop a rock in a lake, it sinks to the bottom. Fire passes upwards to the highest location, so if you make a fire, it always passes upwards through the air.

One conclusion that this led to is that objects which were made of a higher percentage of earth would feel a greater drive to reach their natural location. Since earth is the heaviest of the elements, this would mean that heavier objects would fall faster than lighter objects. Also, they would fall with a natural constant velocity.

That philosophy stood for more than 2000 years until Galileo, in the 1600's made a series of arguments, and conducted a series of experiments, that proved that neither of those two conclusions was accurate. He showed that the natural tendency of all unsupported objects is to fall towards the center of the earth with the same acceleration: $9.8m/s^2$. That number $9.8m/s^2$ is used so often that it as given its own symbol: "g". In modern terms, his conclusion can be stated as follows.

All unsupported objects fall towards the center of the earth with an acceleration of g: $9.8m/s^2$.

This statement requires some explanation and some caveats.

1. Unsupported means that nothing is holding the object up. So if you release something and nothing is stopping it from falling, then it is unsupported. In that case, all objects will experience the same acceleration downwards. It does not depend on how heavy the object is: all objects fall with that same acceleration.

2. Support can also come from air resistance. So a parachute provides support by catching air that slows down the sky diver. In that case, the parachutist is not an unsupported object: he or she is supported by air resistance. But this is generally true to a lesser extent. So a feather or an uncrumpled piece of paper also receives support from the air: so they don't fall with a constant acceleration either. Galileo's conclusion is an idealization, it assumes that we can ignore air resistance, which is never completely true near the earth (or airplanes and parachutes would have a hard time of it) but will work for the problems we'll be doing.

3. His conclusion **does not** depend on the motion of the object. So baseballs thrown toward home plate, dropped, or thrown straight up all fall with the same acceleration towards the center of the earth. This is an area of great confusion for students, so you'll be reminded of it often. Whenever nothing is stopping an object from falling, it will accelerate downwards at $9.8m/s^2$, regardless of its overall motion.

In this book, we will assume that air resistance can be ignored unless it is specifically stated to be a factor.

Example 13
An object is dropped near the surface of the earth. What will its velocity be after it has fallen for 6.0s?

$v = ?$
$v_o = 0$
$t = 6.0s$
$a = g = -9.8m/s^2$ — All unsupported objects have an acceleration of $9.8m/s^2$ downwards

$a = \frac{\Delta v}{t}$
$a = \frac{v - v_0}{t}$

Solve for v: First, multiply both sides by t

$at = v - v_o$

Then add v_o to both sides

$v_o + at = v$

Switch so that v is on the left side of the =

$v = v_o + at$

Substitute in values and solve

$v = 0 + (-9.8m/s^2)\ (6.0s)$
$v = -59m/s$

The Kinematics Equations

So far we have two definitions of motion that we will be using the foundation for our study of motion: $v_{avg} \equiv \frac{\Delta x}{t}$ and $a \equiv \frac{\Delta v}{t}$. We need to add just one more equation to complete our foundation; and then we can start building the set of equations that we'll be using to solve a range of problems throughout this book. The last equation tells how to calculate an object's average velocity if we know its initial and final velocity. It turns out that under the condition of constant acceleration an object's average velocity is just the average of its initial and final velocities. This average is computed just by adding the two velocities, v and v_0, together and dividing by 2:

$v_{avg} = \frac{v - v_0}{2}$

Or, since dividing by 2 is the same as multiplying by ½

$v_{avg} = ½\ (v_0 + v)$

This will be true whenever the acceleration is constant. However, that condition of constant acceleration will hold true not only for this course, but for most all the high school or university physics that you will take.

It'd be possible to solve all problems involving the location, velocity and acceleration of an object just using the two definitions and the computation for average velocity shown above. However, in physics it's sometimes best to do some harder work up front so as to make our work easier later on. In this case, we'll use the above three equations to create a set of kinematics equations that are easier to work with. We'll first derive those equations algebraically; then we'll derive them using a graphical approach. Then we'll practice working with them.

Let's start with by using our definition of acceleration to derive an equation that will tell us an object's velocity as a function of time.

$a \equiv \frac{\Delta v}{t}$

Substitute: $\Delta v = v - v_o$

$a = \frac{v - v_0}{t}$

Multiply both sides by t

$at = v - v_o$

Add v_o to both sides

$v_o + at = v$

Rearrange to solve for v

$\mathbf{v = v_o + at}$

This equation tells us that an object's velocity at some later time will be the sum of two terms: its velocity at the beginning of the problem, v_o, and the product of its acceleration, a, and the amount of time it was acceleration, t. If its acceleration is zero, this just says that its velocity never changes. If its acceleration is not zero, this equation tells us that the object's velocity will change more as more time goes by and it will change faster if the size of its acceleration is greater. Often in physics we want to know the final position or velocity of an object after a certain amount of time. The bolded equation above gives us a direct way of calculating velocities at later times given initial conditions: This is a key kinematics equation.

Example 14

What will an object's velocity be at the end of 15s if its initial velocity is -15m/s and it is subject to an acceleration of +4.5m/s²?

$v = ?$
$v_o = -15m/s$
$t = 15s$
$a = +4.5m/s^2$

$v = v_o + at$
$v = -15m/s + (+4.5m/s^2)(15s)$
$v = -15m/s + 68m/s$
$v = +53m/s$

Example 15

How long will it take an object to reach a velocity of 86m/s if its initial velocity is 14m/s and it experiences an acceleration of 1.5m/s?

$v = 86m/s$
$v_o = 14m/s$
$t = ?$
$a = +1.5m/s^2$

$v = v_o + at$

Solve for t: First subtract v_o from both sides

$v - v_o = at$

Then divide both sides by a

$\frac{v-v_0}{a} = t$

Finally switch sides so t is on the left

$t = \frac{v-v_0}{a}$

Now substitute in the values and solve

$t = \frac{86\frac{m}{s}-14\frac{m}{s}}{1.5\frac{m}{s^2}}$

$t = \frac{72\frac{m}{s}}{1.5\frac{m}{s^2}}$

$t = 48s$

Note that just as 72 divided by 14 equals 48, so too does (m/s) / (m/s^2) equal seconds. This can be seen if one remembers that dividing by a fraction is the same a multiplying by its reciprocal: so (m/s) / (m/s^2) is the same as (m/s) (s^2/m). In this case, it can be seen that the meters cancel out as does one of the seconds in the numerator, leaving only seconds in the numerator; which is the correct unit for time.

Example 16
What acceleration must an object experience if it is to attain a velocity of 40m/s to the north in a time of 18s if it's starts out with a velocity of 24m/s towards the south?

For this problem, let's define north as positive and south as negative. Then,

$v = +40$m/s
$v_0 = -24$m/s
$t = 18$s
$a = ?$

$v = v_0 + at$

Solve for t: First subtract v_0 from both sides

$v - v_0 = at$

Then divide both sides by t

$\frac{v-v_0}{t} = a$

Finally switch sides so a is on the left

$a = \frac{v-v_0}{t}$

Note that this is just our original definition for acceleration. We could have just used that definition, but it's easy enough to recover from the equation we'll be using...either way works. Now substitute in the values and solve

$a =) \frac{40\frac{m}{s} - 24\frac{m}{s}}{18s}$

Note that in substituting -24m/s for v_0 we put it into its own parentheses. That's so we don't lose the negative sign...a common mistake. Now we can see that -(-24m/s) equals +24m/s

$a = \frac{64\frac{m}{s}}{18s}$
$a = 3.6$ m/s^2

Since the answer is positive, the acceleration must be towards the north based on our original decision that north was positive

$a = 3.6$ m/s^2 towards the north

Note that just as 64 divided by 18 equals 3.6, so too does (m/s) / s equal m/s^2. This can be seen if one remembers that dividing by a fraction is the same a multiplying by its reciprocal: so (m/s) / s is the same as (m/s) x (1/s).

We now have a useful equation for determining how an object's velocity will vary with time, given its initial velocity and its acceleration. We need to derive a similar expression that will tell us where an object is located as a function of time given its initial position and velocity and its acceleration.

We need to combine three of our equations together to do that

$v = v_o + at$	The equation we just derived from the definition of acceleration
$v_{avg} \equiv \frac{x - x_0}{t}$	The definition of average velocity
$v_{avg} = ½ (v + v_0)$	The equation for average velocity in the case of constant acceleration

Since we have two equations for average velocity, v_{avg}, they must be equal to each other.

$v_{avg} = v_{avg}$

We can then substitute in the two different equations for v_{avg} from above: one on the left side of the equals sign and the other on the right

$[\frac{x - x_0}{t}] = [½ (v + v_0)]$

Let's solve this for x: first multiply both sides by t to get it out of the denominator on the left

$x - x_0 = ½ (v + v_0)t$

Then add x_0 to both sides to get x by itself

$x = x_0 + ½ (v + v_0)t$

Distribute t into the parentheses on the right

$x = x_0 + ½ vt + ½v_0t$

Now substitute in our new equation for v: $v = v_o + at$

$x = x_0 + ½ (v_o + at) t + ½v_0t$

Distribute ½t into the parentheses

$x = x_0 + ½ v_ot + ½at^2 + ½v_0t$

Combine the two ½ v_ot terms

$\mathbf{x = x_0 + v_ot + ½at^2}$

This is another of the key kinematics equations. It allows us to determine where an object will be as time goes by based on a set of initial conditions. In this case, there are three terms: $\mathbf{x_0}$ tells us where the object started; $\mathbf{v_o}t$ tells us how fast it was moving initially and how long it's been traveling; and ½$\mathbf{a}t^2$, tells us how much its acceleration has affected the distance it has traveled. The reason that t in the last term is squared is that not only does it's velocity change more as time goes by, it also has had more time for that change in velocity to affect how far it's gone.

Example 17

A car is at rest when it experiences an acceleration of $2.0 m/s^2$ towards the north for 5.0s. How far will it travel during the time it accelerates?

For this problem, let's define north as positive and south as negative. Also, since we're not told where the car starts, let's just define its initial position as the origin for this problem, zero. In that case, the distance it travels will just be its position, x, at the end of the problem. Then,

$x_0 = 0$

$x = ?$
$v_o = 0$
$t = 5.0s$
$a = 2.0m/s^2$

$x = x_0 + v_ot + ½at^2$

The equation is already solved for x so we just have to substitute in numbers. However, a good first step is to cross out the terms that will clearly be zero, in this case the first and second terms. (Since $v_o = 0$, anything times v_o will also be zero.)

$x = ½at^2$

That vastly simplifies the equation and avoids some algebra mistakes. Now we can substitute numbers in for the variables.

$x = ½(2.0m/s^2)(5.0s)^2$

Make sure to square 5.0s before multiplying it by anything else

$x = (1.0m/s^2)(25s^2)$
$x = 25m$
The car will travel 25m to the north during the time that it accelerates

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Example 18
An object accelerates from rest. How long will it take for it to travel 40m if its acceleration is $4m/s^2$?

Since we're not told where the object starts, let's just define its initial position as the origin for this problem, zero. In that case, the distance it travels, 40m, will just be its position, x, at the end of the problem. Also, since it's initial at rest that means that its initial velocity is zero. Then,
$x_0 = 0$
$x = 40m$
$v_o = 0$
$t = ?$
$a = 2.0m/s^2$

$x = x_0 + v_ot + ½at^2$

Let's cross out the terms that will clearly be zero, in this case the first and second terms. (Since $v_o = 0$, anything times v_o will also be zero.)

$x = ½at^2$

Now let's solve this for t: Multiply both sides by 2 and divide both sides by a

$\frac{2x}{a} = t^2$

Now take the square root of both sides, to get t instead of t^2, and switch t to the left

$t = \sqrt{\frac{2x}{a}}$

Now we can substitute in the values

$t = \sqrt{\frac{2(40m)}{4\frac{m}{s^2}}}$

$t = \sqrt{\frac{80m}{4\frac{m}{s^2}}}$

t = $\sqrt{20s^2}$

t = 4.47s

The first equation that we derived allows us to determine the velocity of an object as a function of time if we know its acceleration. The second equation allows us to determine the position of an object as a function of time if we know its initial position and velocity and its acceleration. Sometimes we use both equations to solve one problem.

Example 19

A plane must reach a speed of 36m/s in order to take off and its maximum acceleration is 3.0m/s. How long a runway does it require?

Solving this problem requires us to use both our kinematics equation. First, let's figure out how much time it must accelerate to reach takeoff velocity. Then, let's figure out how far it will travel in that time.

$x_0 = 0$
x = ?
$v_o = 0$
v = 36m/s
t= ?
a = 3.0m/s^2

$v = v_o + at$

Solve for t

$v - v_o = at$

t = $\frac{v - v_0}{a}$

t = $\frac{(36\frac{m}{s} - 0)}{3\frac{m}{s^2}}$

t = 12s

Now we can add that new piece of information to what we knew before:
$x_0 = 0$
x = ?
$v_o = 0$
v = 120m/s

t = 12s
a = 3.0m/s^2

We can now use the second equation to solve for the location of the plane when it takes off. That will be the minimum required length of the runway.

$x = x_0 + v_o t + \frac{1}{2}at^2$

Eliminating the zero terms

$x = \frac{1}{2}at^2$
$x = \frac{1}{2}(3.0m/s^2)(12s)^2$
$x = (1.5m/s^2)(144s^2)$
$x = 216m$

The final kinematics equation that we'll develop combines the first two so that we can determine the velocity of an object as a function of its position, rather than as a function of time. That would allow us to have solved Example 18 in just one step. Essentially we derive this equation by doing exactly what we did in Example 18, but without inserting in any numbers, we leave everything as variables. The result is a solution that we can use in the future to save a lot of work.

Let's first solve our velocity versus time equation for time.
$v = v_o + at$
$v - v_o = at$
$at = v - v_o$
$t = \frac{v-v_0}{a}$

Then we'll use that expression for time in the equation that tells us an object's position as a function of time. That will eliminate time from that equation.

$x = x_0 + v_o t + \frac{1}{2}at^2$

Now we'll substitute in $[\frac{v-v_0}{a}]$ wherever we see a "t". The brackets just help us see what we've done. Review this next equation carefully to see that wherever there used to be "t" there's now $[\frac{v-v_0}{a}]$.

$x = x_0 + v_o[\frac{v-v_0}{a}] + \frac{1}{2}a(\frac{v-v_0}{a})^2$

Let's subtract x_0 from both sides, distribute the v_o into the second bracket and square the contents of the third bracket.

$x - x_0 = [\frac{v_0 y - v_0{}^2}{a}] + \frac{1}{2}a(v - v_o)^2 / a^2$

We can now cancel one of the a's in the last term and use the fact that
$(v - v_o)^2 = v^2 - 2v\, v_o + v_o{}^2$

$x - x_0 = [\frac{v_0 y - v_0{}^2}{a}] + \frac{1}{2}(v^2 - 2v\, v_o + v_o{}^2)$

Since a is in the denominator of both terms on the right we can simplify a bit by multiply all the terms by a

$a(x - x_0) = v_o v - v_o^2 + \frac{1}{2}(v^2 - 2v\,v_o + v_o^2)$

Now let's distribute the ½ into the last parentheses on the right

$a(x - x_0) = v_o v - v_o^2 + \frac{1}{2}v^2 - v\,v_o + \frac{1}{2}v_o^2$

By combining the two vv_o terms, they cancel out. At the same time we can combine $- v_o^2$ and $\frac{1}{2}v_o^2$ to get $-\frac{1}{2}v_o^2$

$a(x - x_0) = \frac{1}{2}v^2 - \frac{1}{2}v_o^2$

Now multiply both sides by 2 to cancel out the ½'s

$2a(x - x_0) = v^2 - v_o^2$

Switching the terms from left to right completes this derivation

$\mathbf{v^2 - v_o^2 = 2a(x - x_0)}$

This is sometimes written by substituting d for $(x - x_0)$ since that is just the distance that the object has traveled and it's easier to read

$\mathbf{v^2 - v_o^2 = 2ad}$

This equation lets us determine how the velocity of an object will change as its position changes. Having done all this work now, will save us work later. In Example 19, let's take a look at how we would use this equation to solve the same problem as was posed in Example 18

Example 20
As was the case in Example 18, a plane must reach a speed of 36m/s in order to take off and its maximum acceleration is 3.0m/s². How long a runway does it require?

$x_0 = 0$
$x = ?$
$v_o = 0$
$v = 36 m/s$
$a = 3.0 m/s^2$

$v^2 - v_o^2 = 2a(x - x_0)$

Simplify by eliminating the zero terms, v_o & x_0

$v^2 = 2ax$

Solve for x by dividing by 2a and switching sides

$x = \frac{v^2}{2a}$

$x = \frac{36^2}{2(3\frac{m}{s^2})}$

$x = 216m$

Problem Solving with the Kinematics Equations

The Kinematics Equations
$x = x_0 + v_o t + \frac{1}{2}at^2$
$v = v_o + at$
$v^2 - v_o^2 = 2a(x - x_0)$

All kinematics problems can be solved by using either one or two of the above equations. It is never necessary to use all three equations to answer one question. The biggest question students have in working with these equations is which one to use?

First, you need to relax and understand that you can't get the wrong answer by using the wrong equation; you just won't get an answer at all. You'll just find that you're missing the information that you need to solve the problem using that equation. At that point, you should realize that you need to use a different equation, or read the problem again to see if you're missing a piece of information that you need but have overlooked.

For instance, if a problem indicates or implies that an object was at rest at the beginning of the problem that means that its initial velocity was zero. Sometimes this is obvious... sometimes it isn't. For instance, if I "drop" something, the implication is that it had a zero initial velocity, but that isn't explicitly stated; it's implied. Physics will help you learn to read very carefully to understand what the author meant when they wrote the problem or described the situation.

So, the first step in solving the problem is to read it very carefully.

The second step is to read it again. This time writing down the information you've been given in terms of the variables with which you'll be working. For instance, translating "dropped" into "$v_o = 0$". One of the pieces of information that you'll be given is what you're supposed to be looking for: what's the question. If the author asks "What is its final velocity?" that is translated as "$v =?$" and becomes another of the facts to add to your list of facts that you'll use to solve the problem.

The next step is to determine which of the kinematics equations relate your collection of facts to one another. Each equation represents a relationship between a different set of facts: picking the correct equation is just a matter of determining which equation relates to this specific set of facts. If you pick the wrong one, no harm will be done (except some wasted time) since you'll find you just don't have the right facts to use that equation.

If the problem that you're working with doesn't have time as one of its facts, then you'll be using the third of the equations listed above: $v^2 - v_o^2 = 2a(x - x_0)$; it's the only one that doesn't include time as a factor. If time is included, you'll be using one of the first two. In that case, you just need to determine which of those first two equations to use. If the problem deals with the position of the object as a function of time, then you'd use the first equation: $x = x_0 + v_o t + \frac{1}{2}at^2$. If it is dealing with the velocity of the object as time goes by, you'll use the second equation: $v = v_o + at$. It's really as simple as that.

Example 21
In this example, we're just going to decide which equation(s) will be needed to solve each problem.

1. A ball is subject to an acceleration of -9.8 m/s^2. How long after it is dropped does it take to reach a velocity of -24m/s?
2. A ball is released from rest and subject to an acceleration of -9.8 m/s^2. How far will it travel before it reaches a velocity of 24m/s?
3. A dropped ball is subject to an acceleration of -9.8 m/s^2. How far will it travel in the first 5.0s?
4. You throw an object upwards from the ground with a velocity of 20m/s and it is subject to a downward acceleration of 9.8 m/s^2. How high does it go?
5. You throw an object upwards from the ground with a velocity of 20m/s and it is subject to a downward acceleration of -9.8 m/s^2. How much later does it momentarily coming to a stop?
6. You throw an object upwards from the ground with a velocity of 20m/s and it is subject to an downward acceleration of 9.8 m/s^2. How high is it after 2.0s?

Take a second to write down the facts in each problem and then determine which equation you would use. Then compare your results to those shown below.

1. "an acceleration of -9.8 m/s^2" means $a = -9.8$ m/s^2
"How long after" means $t = ?$
"it is dropped" means $v_o = 0$
"to reach a velocity of -24m/s" means $v = -24$m/s

Since time, t, is a factor, we need to only choose between the first two equations. Since velocity, v, is a factor, it must be the second equation:

$$\mathbf{v = v_o + at}$$

2. "released from rest" means $v_o = 0$
"an acceleration of -9.8 m/s^2" means $a = -9.8$ m/s^2
"How far will it travel" means $x_0 = 0$ and $x = ?$
"reaches a velocity of 24m/s" means $v = -24$m/s

Since time, t, is <u>not</u> a factor, we need to use the third equation

$$\mathbf{v^2 - v_o^2 = 2a(x - x_0)}$$

3. "A dropped ball" means $v_o = 0$
"an acceleration of -9.8 m/s^2" means $a = -9.8$ m/s^2
"How far will it travel" means $x_0 = 0$ and $x = ?$
"in the first 5.0s" means $t = 5.0$s

Since time is a factor, we need to only choose between the first two equations. Since position, x, is a factor, it must be the first equation:

$$\mathbf{x = x_0 + v_o t + \tfrac{1}{2}at^2}$$

4. "upwards from the ground with a velocity of 20m/s" means $v_o = +20$m/s and $x_0 = 0$
"a downward acceleration of

9.8 m/s^2"	means $a = -9.8\ m/s^2$
"How high does it go?"	means $x = ?$ and $v = 0$ (the second fact, $v = 0$, is less obvious but is implied by the fact that when it reaches it's greatest height it must momentarily stop...or it would go higher

Since time, t, is <u>not</u> a factor, we need to use the third equation

$$\mathbf{v^2 - v_o^2 = 2a(x - x_0)}$$

5. | | |
   |---|---|
   | "upwards from the ground with a velocity of 20m/s" | means $v_o = +\ 20m/s$ and $x_0 = 0$ |
   | "a downward acceleration of 9.8 m/s^2" | means $a = -9.8\ m/s^2$ |
   | "How much later" | means $t = ?$ |
   | "momentarily coming to a stop" | means $v = 0$ |

   Since time, t, is a factor, we need to only choose between the first two equations. Since velocity, v, is a factor, it must be the second equation:

   $$\mathbf{v = v_o + at}$$

6. | | |
   |---|---|
   | "upwards from the ground with a velocity of 20m/s" | means $v_o = +\ 20m/s$ and $x_0 = 0$ |
   | "a downward acceleration of 9.8 m/s^2" | means $a = -9.8\ m/s^2$ |
   | "How high is it" | means $x = ?$ |
   | "after 2.0s" | means $t = 2.0s$ |

   Since time is a factor, we need to only choose between the first two equations. Since position, x, is a factor, it must be the first equation:

   $$\mathbf{x = x_0 + v_ot + \tfrac{1}{2}at^2}$$

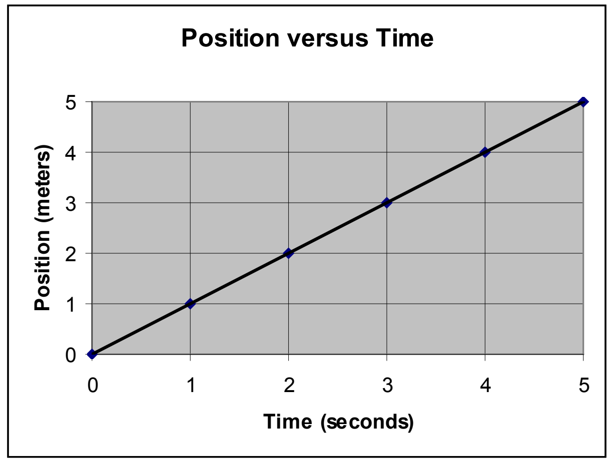
Interpreting Motion Graphs

There are two types of motion graphs that we'll be considering, "Position versus Time" and "Velocity versus Time". In both cases the horizontal axis, the x- axis, is used to record the time. In a Position versus Time graphs, the vertical axis, the y-axis, is used to record the position of the object. In a Velocity versus Time graph, the vertical axis, the y-axis, is used to record the object's velocity. In this section, we're going to learn how to make and interpret these graphs and see their relationship to each other.

Position versus time graphs for Constant Velocity

If while you were moving you were to record your position each second, it'd be easy to make a position versus time graph. Let's take the case that you're walking away from your house with a constant velocity of +1m/s (note that since velocity is a vector defining it requires a direction, "+", and a magnitude, "1m/s"). If you define your house as "zero" and define your starting time as zero then the first five seconds of your walk would give you the following data.

Time (t) seconds	Position (x) meters
0	0
1	1
2	2
3	3
4	4
5	5



To create a position versus time graph all you need to do is graph those points and then connect them with a straight line.

This graph allows you to directly read your position at any given time. In fact, it allows you to determine your position for times at which you made no measurement, that's the meaning of the line connecting the points. For instance, your position at 1.5 seconds can be seen to be at x = 1.5m. Now this assumes that you were traveling at a constant velocity, but that's the assumption that was made in creating this graph.

You can also indirectly read your velocity from this chart. **The velocity of an object will be the slope of the line on its Position versus Time graph.**

The definition of the slope of a line, m, is m $\equiv \frac{\Delta v}{\Delta x}$. This means that the slope of a line is determined by how much the vertical value, the y-value, of the line changes for a given change in its horizontal value, its x-value. If it doesn't change at all, then the line is horizontal, it has no tilt. If it has a positive value it is sloping upwards, since that means that the y-coordinate gets larger as you move to the right along the x-axis. A negative slope means that the line is tilted downwards, since its y-value decreases as you move to the right.

If you make a graph of the position versus time for a moving object, with position shown on the y-axis and time on the x-axis, then the slope of that line is given by m $\equiv \frac{\Delta y}{\Delta x}$, but in this case the y-values are the position, x, and the x-values are time, t.

> This can be confusing since the "x" in the definition of slope is different than the "x" used in kinematics equations. In the definition of slope, x means the horizontal axis. In the discussion of motion, x means the position of the object. When we graph the position of an object versus time, we always put the position on the vertical, y-axis, and time, t, on the x-axis. This can lead to confusion since the y-coordinate on the position versus time graph gives the position, which is x in kinematics equations.

So the slope of a line on that graph becomes:

$m \equiv \frac{\Delta y}{\Delta x}$ — but since the y-values represent the position, x, and the x-values give the time, t, this becomes

$m \equiv \frac{\Delta x}{\Delta t}$ — but the definition for velocity is the same, v ≡ Δx / Δt, therefore

m = v — the slope of a line show on the Position versus Time graph give us its velocity

For the graph shown above, the slope of the line is:

$m \equiv \frac{\Delta y}{\Delta x}$ — using the first and last point (any pair of points will work)

$m = \frac{5m-0m}{5s-0s}$

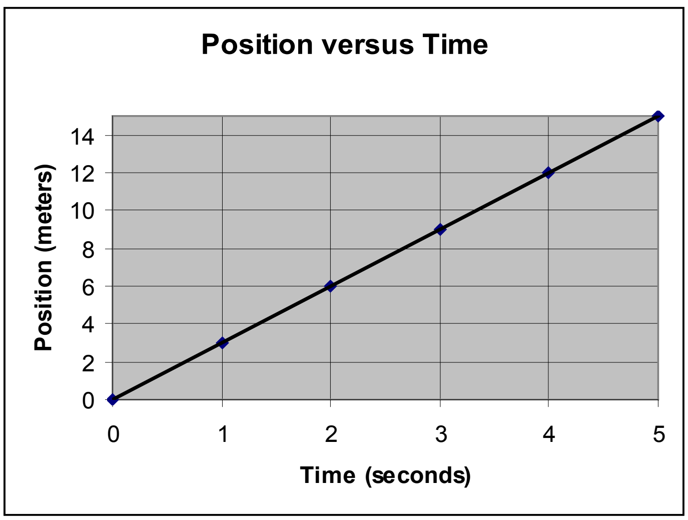
$m = \frac{5m}{5s}$

m = 1 m/s

v = 1 m/s

Example 22

Determine the location and velocity of the object in the below graph at when t = 2.5s.



Solution: The location of the object can be read directly off the graph by noting that when the time is equal to 2.5s then the position is equal to 7 meters. The velocity is constant during the entire trip (hence the straight line) and so it can be found by determining the slope of that line between any two points. Usually we pick points that are easy to read and are as far apart as possible. In this case, let's use the points (0,0) and (4,12).

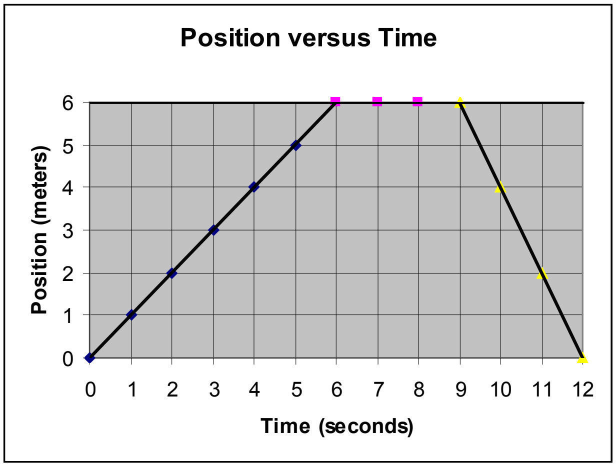
$m \equiv \frac{\Delta y}{\Delta x}$

$m = \frac{\Delta 12m-0m}{4s-0s}$

$m = \frac{12m}{4s}$

m = 3 m/s

v = 3 m/s



Now it's possible that the velocity of an object will change during the time it is being observed. For instance, if you were to walk away from your house at a velocity of 1m/s for 6 seconds, stop for 3 seconds and then run back to your house in 3 seconds, the Position versus Time graph for your trip would look like this.

You can read this graph to determine your position at any time during the trip. You can also see from this that during your trip you had three different velocities.

Your initial velocity is given by the slope of the line during the first 6 seconds.

$m \equiv \frac{\Delta y}{\Delta x}$

$m = \frac{6m - 0m}{6s - 0s}$

$m = \frac{6m}{6s}$

m = 1 m/s

v = 1 m/s

During the time that you're standing still, your velocity must be zero. The slope of the line must be zero if your velocity is zero, so this is the flat portion of the curve between 6 and 9 seconds. Just for completeness, you get the same results analytically.

$m \equiv \frac{\Delta y}{\Delta x}$

$m = \frac{6m - 6m}{9s - 6s}$

$m = \frac{0m}{6s}$

m = 0 m/s

v = 0 m/s

Finally, during your return trip the slope of the line is negative, meaning that you have a negative velocity.

$m \equiv \frac{\Delta y}{\Delta x}$

$m = \frac{0m - 6m}{12s - 9s}$

$m = \frac{-6m}{3s}$

m = -2 m/s

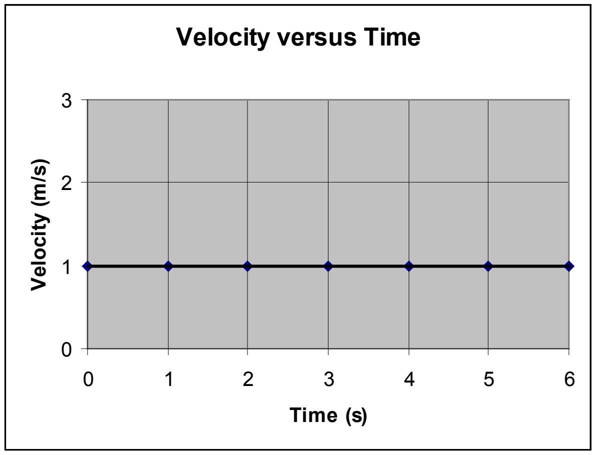
v = -2 m/s

Velocity versus Time graphs for Constant Velocity

Any motion that can be recorded using a Position versus Time graph can also be recorded using a

Velocity versus Time graph. Your choice of graph will have different benefits, but it's important that you be able to see how they relate to one another.

Let's take the first Position versus Time graph that we did above and recast it as a Velocity versus Time graph. In this case, the vertical axis records velocity; the horizontal axis continues to indicate the time. In the first graph, you maintained a constant velocity of 1 m/s for 6 seconds so this becomes:



In this case, the object's velocity can be read directly off the graph, but its displacement and the distance it has traveled cannot. However, we can determine the object's displacement, how far it is from where it started, and the distance it has traveled by measuring the area under the curve (in this case the horizontal line at v = 1 m/s). (If the velocity is always positive, the distance traveled and displacement will be the same.)

A rectangle has two pairs of opposing sides. In this case, one side will be the horizontal line indicating the velocity that the object is traveling and the side opposing that is the horizontal axis. The second pair of sides is a vertical line drawn straight up from the time we start measuring and the side opposing that is a vertical line drawn straight up from the time we stop measuring.

So for an object moving at constant velocity we can define a rectangular shape whose height is its velocity and whose length is the time interval that we're studying. The area of a rectangle is given by its height multiplied by its length, so the area of this rectangle is it's velocity, v, multiplied by the elapsed time, t.

Area = (height)(length)
A = (velocity)(time)
A = vt

But early in the chapter we determined that $\Delta x = vt$, so

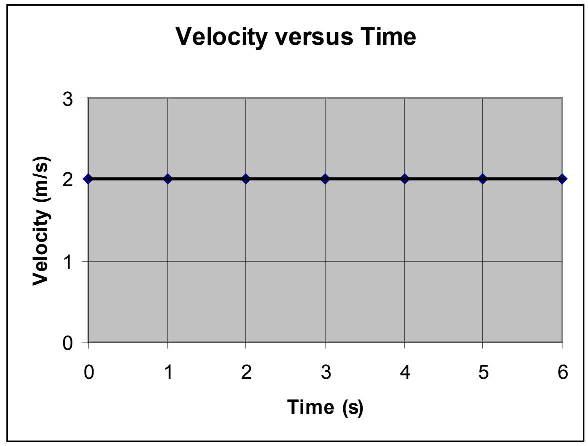
A = Δx

The displacement of an object is determined by the area between its velocity graph and the horizontal axis.

In the current example the horizontal sides are the horizontal line at v = 1 m/s and the horizontal axis, and the vertical sides are formed by the vertical axis and t = 6s. The dimensions of that rectangle are 1m/s tall by 6s long. So the object's displacement is simply the product of those two, or 6m. This is the same result you would expect, an object moving at a constant velocity of 1 m/s for 6s would be displaced 6m. It has also traveled a distance of 6m since no negative motion was involved

If the displacement is only positive, it is equal to the distance that the object has traveled. While we derived this for constant velocity, it will always be true that the displacement will be equal to the area under the Velocity versus Time curve and that, if the velocity is always positive, that the distance traveled will equal the displacement of the object.

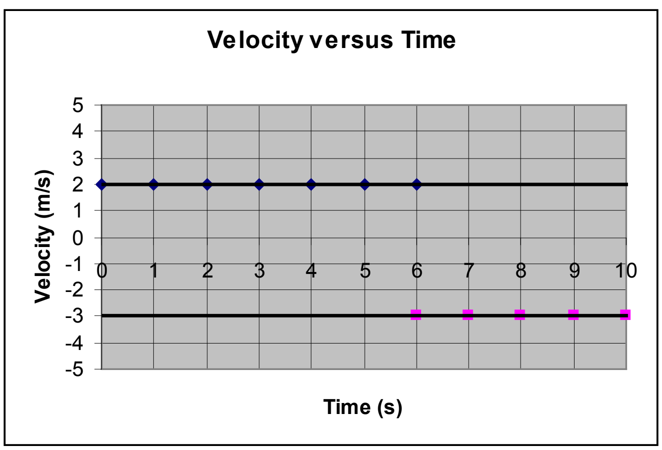
Example 23
Determine the displacement of the following object, and the distance it has traveled, during its first 3 seconds of travel.



Solution: The area under the curve of its Velocity versus Time graph is its displacement. Since we are only considering the first 3 seconds of its travel, the height is 2 m/s and the length is 3 seconds, so its displacement is 6 meters. This is also equal to the distance it has traveled.

In the case that an object has both a negative and positive velocity, the distance it has traveled and its displacement will not be the same. This is because distance does not depend on direction while displacement does.

Example 24
Determine the distance traveled and the displacement of the following object during its entire trip. It travels at a velocity of +2m/s for the first six seconds and then a velocity of -3m/s during the last four seconds.



The object's displacement during the first six seconds is given by:
Δx = Area
$\Delta x = (+2m/s)(6s)$
$\Delta x = 12m$

During the last four seconds of its trip its displacement is
Δx = Area
Δx = (-3m/s)(4s)
Δx = -12m

So its total displacement is the sum of those two contributions:
Δx = 12m + (-12m)
Δx = 0m.

On the other hand, the total distance it traveled is given by the sum of the two areas, treating both as positive numbers since distance traveled can never be negative. So the distance traveled is the sum of 12m and 12m, or 24m.

d = 12m + 12m
d = 24m

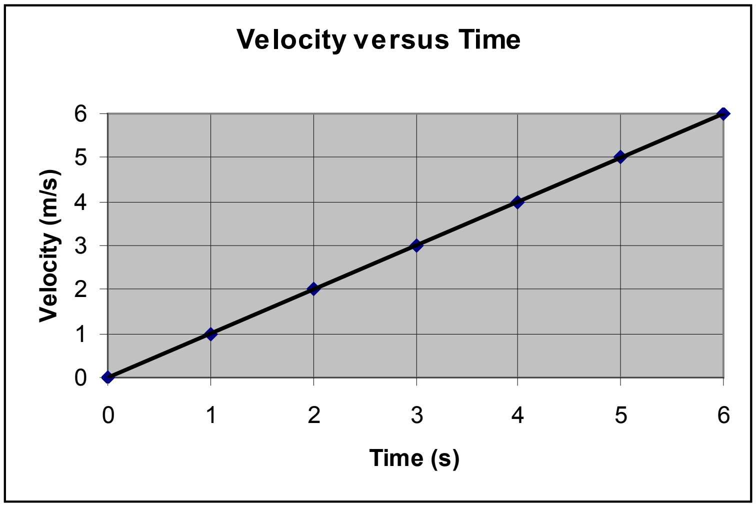
If we take motion to the right as positive and motion to the left as negative, the physical interpretation of this is that the object traveled 12 m to the right, momentarily stopped and then traveled 12m to the left. It moved a distance of 24m, but ends up at the origin.

Velocity versus Time graphs for Constant Acceleration
So far we have only considered motion at constant velocity. However, the same principles apply to motion at constant acceleration. If an object's velocity is changing with time, it's Velocity versus Time graph will have a slope. That slope will give its acceleration.

The slope of a line on the Velocity versus Time graph is:
m $\equiv \frac{\Delta y}{\Delta x}$ — but since the y-values represent the velocity, v, and the x-values give the time, t, this becomes
m $\equiv \frac{\Delta v}{\Delta t}$ — but the definition for acceleration is the same, a $\equiv \frac{\Delta v}{\Delta t}$, therefore
m = a

The slope of a line on the Velocity versus Time graph for an object gives us its acceleration.

Let's calculate the acceleration of the following object.



For the graph shown above, the slope of the line is:

$m \equiv \frac{\Delta y}{\Delta x}$ Now use the first and last point (any pair of points will work)

$m = \frac{6\frac{m}{s} - 0\frac{m}{s}}{6s - 0s}$

$m = \frac{6\frac{m}{s}}{6s}$

m = 1 m/s^2

a = 1 m/s^2

It is still the case that the area under the Velocity versus Time graph will give us the displacement and the distance traveled. However, the shape is not longer a rectangle, it's a triangle. The area of a triangle is given by the formula: Area = ½ (base) (height). We can use this to determine the displacement and distance traveled; in this case they will be the same since all motion is positive. The base is given by the elapsed time and the height is the greatest velocity attained, since that's the highest point for the triangle. Let's determine the object's displacement during its six seconds of travel.

Δx = Area

Δx = ½ base x height

Δx = ½ vt where v is the velocity at time t

Δx = ½ (+6m/s)(6s)

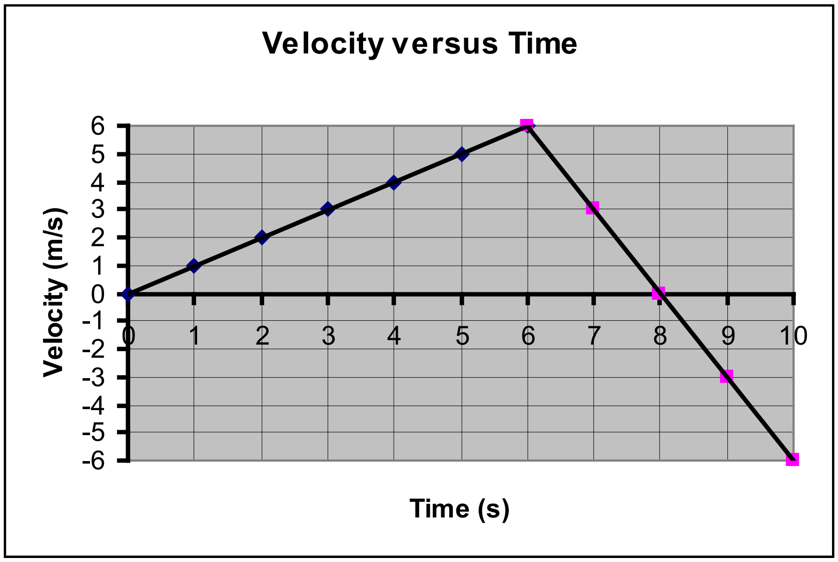
Δx = 18m since all motion is at positive velocity, this is also the distance traveled

d = 18m

In the following example we need to divide the motion into two triangles, since one will have a positive area, below the horizontal axis, and one will have a negative area. That will give us different answers for displacement and distance.

Example 25

Determine the displacement and distance traveled by the object whose motion is described in the graph to the right.



Solution: Since the motion included both negative and positive velocity, we need to separate out those two parts.

For the motion with positive velocity we need to determine the area of the triangle formed above the horizontal axis:

Δx = Area
Δx = ½ base x height
Δx = ½ vt   where v is the maximum positive velocity and t is the total time that the object moved with a positive velocity
Δx = ½ (+6m/s)(8s)
Δx = 24m   This is the displacement due to positive velocity

For the motion with negative velocity we need to determine the area of the triangle formed below the horizontal axis:

Δx = Area
Δx = ½ base x height
Δx = ½ vt   where v is the maximum negative velocity and t is the total time that the object moved with a negative velocity
Δx = ½ (-6m/s)(2s)
Δx = -6m   This is the displacement due to negative velocity

In this case the displacement and distance traveled will be different. The time spent traveling with a negative velocity will reduce the displacement, but will add to the total distance the object moved during its trip.
Δx = +24m + (-6m)
Δx = +18m

To find the distance traveled we just treat the areas of both triangles as positive since distance is never negative.
d = 24m + 6m   since distance is always positive
d = 30m
So after traveling a total distance of 30m, the object is 18m to the right of where it started.

Alternative Derivation of the First Kinematics Equation

We used a lot of algebra to derive the following equation earlier in this chapter:

$$\mathbf{x = x_0 + v_0 t + \tfrac{1}{2}at^2}$$

However, this same equation was also derived graphically in the 1400's by Oresme using the approaches that we just developed: recognizing that the displacement of an object is given by the area under the Velocity versus Time curve.

For those who prefer a great picture to some complicated algebra, it's worth seeing how he did it. We'll also use a similar approach to derive some equations in some later chapters.

All we have to do is leave variables in our graphs instead of numbers. For instance let's start out with an object moving at a constant velocity v_0 for a time t. That means that the graph of its velocity versus time will form a rectangle whose height is v_0 and whose length is t. The area of that rectangle, will give its displacement.

Displacement due to initial velocity

Δx = Area

Δx = height x length

$\mathbf{\Delta x = v_0 t}$ **Due to initial velocity alone**

Now let's add onto that the displacement due to a constant acceleration. If the acceleration is positive, that means it gets a bigger velocity as time goes by and you get the graph shown below (assuming for simplicity that $v_0 = 0$.). We showed earlier that the area under this curve is equal to an object's displacement. In this case, its greatest velocity will be the height of the triangle and the base of the triangle will be the time of its acceleration.

Displacement due to acceleration

Δx = Area

Δx = ½ base x height

$\Delta x = \frac{1}{2} vt$ But recall that that $v = v_0 + at$. In this case, v_0 is zero, so $v = at$. Substituting that in for v we get

$\Delta x = \frac{1}{2} (at)t$

$\mathbf{\Delta x = \tfrac{1}{2} at^2}$ **Due to acceleration alone**

Total Displacement

An object that has an initial velocity and experiences acceleration has a displacement due to both of these terms. So its total displacement will be

$\Delta x = v_0 t + \frac{1}{2} at^2$

But recall that $\Delta x = x - x_0$, so

$x - x_0 = v_0 t + \frac{1}{2} at^2$ Solving for x yields our kinematics equation

$\mathbf{x = x_0 + v_0 t + \tfrac{1}{2} at^2}$

The position of an object at any time will be given by three terms: where it started, x_0, how far it moved due to its initial velocity, $v_0 t$, and how far it moved due to its acceleration, $\frac{1}{2} at^2$. Adding those terms together gets you the same result we got using algebra earlier in the chapter: $\mathbf{x = x_0 + v_0 t + \tfrac{1}{2}at^2}$**.**

