Simple Harmonic Motion, Waves, and Uniform Circular Motion

Introduction

The three topics: Simple Harmonic Motion (SHM), Waves and Uniform Circular Motion (UCM) are deeply connected. Much of what we learned about UCM can be directly applied to SHM…and much of what we will learn about SHM will be directly applied to our study of waves. By making these connections clear, it is possible to get a richer understanding more efficiently than treating them independently. First, let's quickly summarize what we learned about uniform circular motion.

Objects that are traveling along a circular path at constant speed are in uniform circular motion. While their speed is constant, their velocity is constantly changing since their direction is always changing. They are also undergoing constant acceleration. Otherwise, they would travel in a straight line, not in a circle.

Their acceleration and force is directed towards the center of their circular path while their velocity is tangent to that path. The magnitude of their acceleration is given by $a = \frac{v^2}{r}$.

The time it takes to complete a complete circle is called the Period (T) and is measured in units of time, often seconds. If in t seconds an object completes n complete circles, its period is given by $T = \frac{t}{n}$. The number of complete circles completed per unit time is called the Frequency (f) and is measured in units of one over time, often 1/s: Hertz (Hz). If an object completes n complete circles in a time t, its frequency is given by: $f = \frac{n}{t}$. Comparing the formulas for frequency and period we can see that they are inversely related: $T = \frac{1}{f}$ and $f = \frac{1}{T}$.

The velocity of the object is given by $v = \frac{2\pi r}{T}$, where r is the radius of motion. This follows from the fact that the circumference of a circle is given by C = 2πr and is the distance traveled in one revolution; T is the time needed to travel that distance. Since $f = \frac{1}{T}$, this can also be seen to be **v = 2πrf**.

Combining these expression for velocity with our previous expression for acceleration ($a = \frac{v^2}{r}$) we can derive another expression for the magnitude of the acceleration of object undergoing UCM.

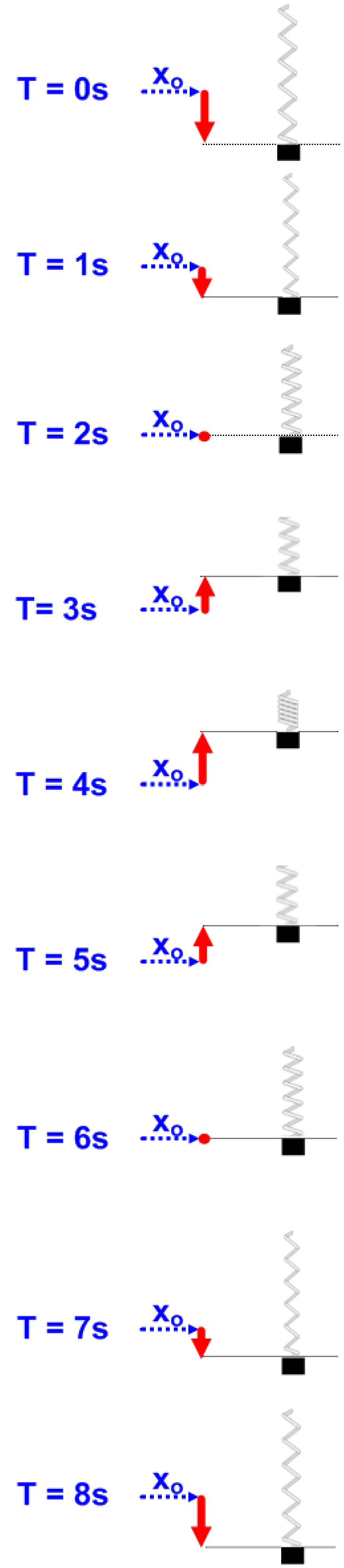
In terms of period:	In terms of frequency:
$a = \frac{v^2}{r}$	$a = \frac{v^2}{r}$
$a = \frac{\left(\frac{2\pi r}{T}\right)^2}{r}$	$a = \frac{(2\pi r f)^2}{r}$
$a = \frac{4\pi^2}{T^2}$	$a = 4\pi^2 r f^2$

All these results will be directly useable for simple harmonic motion once we show you how UCM and SHM are connected.

The Mass-Spring System

Simple harmonic motion is characterized by an object moving repeatedly back and forth through the same path. An example that illustrates this type of motion is the up and down motion of a mass hanging from a spring. If the mass is slowly lowered, the force exerted by the spring upwards ($F_{spring} = -kx$) will increase until it is equal and opposite to the force of gravity. At that location, the equilibrium point of the system, the mass will remain stationary unless another force acts on it.

If you then pull the mass down, it will feel a net force upwards, in the direction opposite to the displacement you've given it. That's due to the spring force increasing in proportion to the distance it's stretched from its initial equilibrium while the force of gravity remains unchanged. The farthest distance that you pull the mass from its equilibrium point is called the **Amplitude** (**A**) of the system.
The diagram to the right shows simple harmonic motion for a mass-spring system which completes one full cycle of motion in 8s. The elapsed time is shown to the left and the equilibrium point is shown by the arrow labeled, x_0, which points to the right. The displacement vector (the vertical arrow) indicates the distance from equilibrium to the top of the mass.



When the mass is released at t=0 it accelerates upward due to the net force acting on it; since the spring is stretched beyond equilibrium the force of the spring upward is greater than the force of gravity downward.

It takes 2s for the mass to reach the equilibrium point. At that location, there is no net force acting on the mass, so its acceleration is zero and it continues through equilibrium with constant velocity: the velocity it picked up from being accelerated back towards equilibrium during its first 2s of its motion.

As it rises above the equilibrium point the upward force of the spring diminishes, but the force on the mass due to gravity stays constant. Therefore, the mass feels a net force, and a resulting acceleration, downward, opposing its velocity. When the mass is at a height equal to its amplitude, but in this case above the equilibrium point, it comes to a momentary stop; this occurs at t=4s. At that point, its acceleration is as large as it was at t=0s, but in the opposite direction.

The mass then accelerates downwards such that it passes through the equilibrium point again at t=6s. Once again, there is no net force acting on the mass at the equilibrium point, but it has a constant velocity...in this case, downwards.

That velocity carries the mass down until it reaches a distance from equilibrium equal to its amplitude of motion at t=8s. It momentarily stops before beginning another cycle. The drawing at t=8s is exactly identical to what it was at t=0s, so the system is reset to its initial condition.

This motion then repeats over and over. If there were no losses due to friction with the air, the heating of the spring through its being bent repeatedly, etc. this motion would continue forever.

SHM and UCM

Let's now see how this is similar to uniform circular motion so we can connect our prior learning about UCM to this new topic of simple harmonic motion (SHM). This will enable us to use the terminology and results we obtained in our study of UCM directly to SHM.

The key is to look at only the up and down motion of an object going in a vertical circle, ignoring its side to side motion. It's as if a powerful light shone on the object from the side and you looked at its shadow moving on a wall. The shadow would just move up and down, its side to side motion would be invisible.

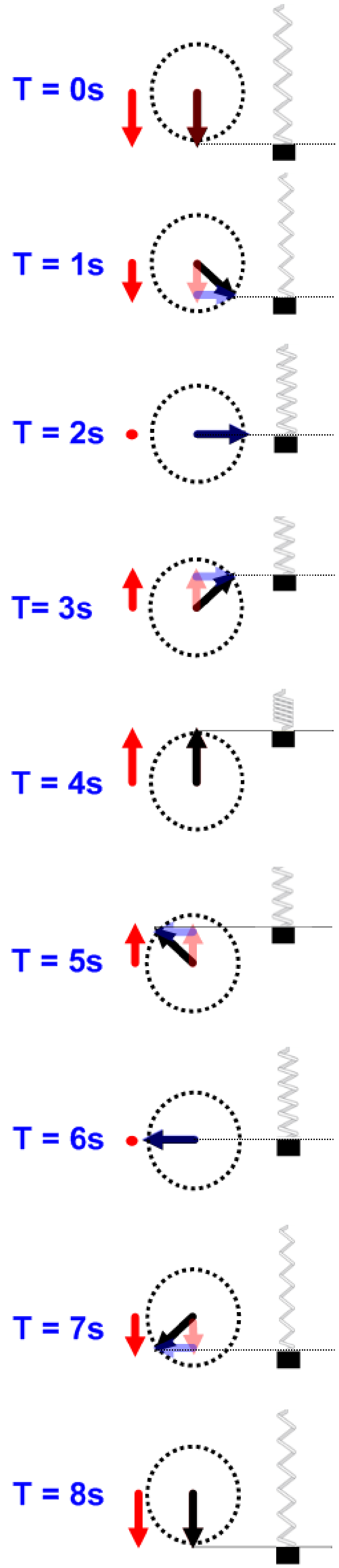
Examine the drawing to the left. Within the circle we show an arrow, a displacement vector, which is drawn from the center of the circle to the object traveling around the circle. We then also show one vector that shows the side to side motion and another that shows the up and down, vertical, motion. The vertical component is also shown to the left of the diagram just to make that component of motion clear.

Next to the object in circular motion we also show a mass-spring system. Its displacement vector, drawn from the equilibrium point to the top of the mass, is also illustrated by that same vertical arrow to the left of both diagrams. In this case, both objects complete one cycle in 8s.

As the object travels in a circle, the vertical component of its displacement from the origin goes from its maximum negative amount at t=0s; becomes zero at t=2s, reaches its maximum positive value at t=4s; becomes zero again at t=6s; and is back at its maximum negative value at T=8s.

Just as the object's motion has a period of 8s; the vertical components of its motion has that same period of 8s. Also, it can be seen that its maximum displacement in the vertical direction, from the origin, will also be the radius of the circle.

But the vertical component of the object's circular motion is identical to the displacement vector for the mass undergoing simple harmonic motion; there is no difference. The mass spring system undergoes an up and down motion that is



identical with the up and down motion of the object in circular motion, the only difference is the lack of a horizontal component of motion in the case of SHM.

The result of this connection is that everything we learned about objects in uniform circular motion will be true for objects in simple harmonic motion.

<u>Translating from UCM to SHM</u>

The Period (T) is identical: $T = \frac{t}{n}$

The frequency (f) is identical: $f = \frac{n}{t}$

Also then: $T = \frac{1}{f}$ and $f = \frac{1}{T}$

Adjustments need to be made in order to translate the UCM equations for amplitude, velocity and acceleration into SHM equations. First, we need to note that while those are constant when looking at the motion of an object traveling in a circle, they are not constant when only looking at one component of its motion. For instance, while the speed of an object in UCM is constant, its vertical speed is not; sometimes its speed is carrying the object up and down, sometimes it's carrying it from side to side (in which case its vertical speed is zero), most of the times it's a mix of the two; the same is true for the object's distance from the center of the circle and for its acceleration. The equations for the displacement, velocity and acceleration of an object in UCM become equations for its **maximum displacement** (**A**), **maximum velocity** (**v_{max}**) and **maximum acceleration** (**a_{max}**) of its vertical motion.

The Amplitude (A) of the vertical motion of an object in circular motion is just the radius of the circle (r) since that's the farthest the object can ever be from the circles center in any dimension: A = r; but its distance (x) from equilibrium will vary from +A to –A, passing through zero. Similarly, the velocity of the object will vary from $+v_{max}$ to $-v_{max}$, passing through zero, and its acceleration will vary from $+a_{max}$ to $-a_{max}$, passing through zero.

Translating from UCM to SHM: **A = r**

$v = \frac{2\pi r}{T}$ from UCM becomes, in SHM: $\boldsymbol{v_{max} = \frac{2\pi A}{T}}$

$v = 2\pi rf$ from UCM becomes, in SHM: $\boldsymbol{v_{max} = 2\pi Af}$

$a = \frac{4\pi^2 r}{T^2}$ from UCM becomes, in SHM: $\boldsymbol{a_{max} = \frac{4\pi^2 A}{T^2}}$

$a = 4\pi^2 rf^2$ from UCM becomes, in SHM: $\boldsymbol{a_{max} = 4\pi^2 Af^2}$

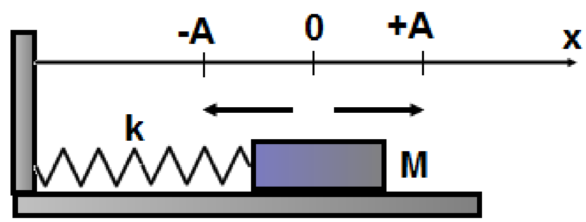
t	x	v	a
0s	-A	0	$+a_{max}$
2s	0	$+v_{max}$	0
4s	+A	0	$-a_{max}$
6s	0	$-v_{max}$	0
8s	-A	0	$+a_{max}$

The relationship between the position of the object and its velocity and acceleration is shown in this chart. While we reference times for the motion illustrated above, that is only so you can reference the diagram; the relationships between the position, velocity and acceleration hold for all SHM motion with any period.

Note that the chart uses the convention that the positive direction is up.

Horizontal Mass-Spring Systems

Our previously drawn conclusions are equally applicable to horizontal or vertical mass-spring systems. In the horizontal system the restoring force in one direction is due to the stretching of the spring and in the opposite direction it is due to the compression of the spring. In theory, this system is much easier to understand since the effect of gravity is not involved. In practice it's very difficult to demonstrate the horizontal design since it requires ideal springs which are symmetrical with regard to the force they exert when either stretched or compressed and it requires a frictionless surface for the mass to slide on. So you will see many more vertical designs demonstrated than horizontal designs.



While those represent practical problems, in theory all the results we derived for a vertical mass-spring system are directly applicable to horizontal designs like this. And these designs are often used for setting up problems as they are theoretically simpler than vertical designs.

Example 1: In the above diagram, a block with a mass M is attached to a spring with a spring constant k. The block undergoes SHM. Where is the block located when:

a) Its speed is a maximum?
b) Its speed is zero?
c) Its acceleration is zero?
d) The magnitude of the net force on the mass is a maximum?

Answers:
a) When x=0 the magnitude of the object's velocity will be the greatest since all the energy is KE at that location.
b) When x=+A and x=-A the object momentarily comes to a stop, all the energy is U_{spring}.
c) When x= 0 the net force on the block is zero, so its acceleration is as well.
d) When x = +A or –A the spring is stretched to its maximum so the net force on the mass is a maximum.

The Period of a Mass-Spring System

We learned earlier that the energy stored in a spring is given by $U_{spring} = ½kx^2$ and that the energy of a moving mass is $KE = ½mv^2$. The total energy in a mass-spring system is put into the system at the beginning from the outside, for instance by pulling the mass down from its equilibrium point. Once the mass is released, the total energy stays constant, but it changes from one form to another. This is always the case with SHM, but in this case, the two forms of energy are U_{spring} and KE.

$$U_{total} = Constant$$

$$U_{total} = U_{spring} + KE$$

$$U_{total} = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

When the object is at +A or –A all the energy is in the spring since it is stretched to its maximum and the object is momentarily at rest. From this we can conclude that $\mathbf{U_{total} = \frac{1}{2}kA^2}$. Similarly, as the mass moves through equilibrium there is no potential energy in the spring, since it is unstretched, so all the energy is in the kinetic energy of the mass; it must be moving with its maximum velocity (v_{max}) at equilibrium and $\mathbf{U_{total} = \frac{1}{2}mv^2_{max}}$. Since the total energy is the constant this means that:

$U_{total} = U_{total}$

$$\frac{1}{2}mv^2{}_{max} = \frac{1}{2}kA^2$$

$mv^2_{max} = kA^2$

$$v^2{}_{max} = \frac{k}{mA^2}$$

$$v_{max} = A\sqrt{\frac{k}{m}}$$

We can now combine this result with our prior result ($v_{max} = 2\pi A/T$) to find the period of motion of a mass-spring system.

$$v_{max} = A\sqrt{\frac{k}{m}} \quad \text{and} \quad v_{max} = \frac{2\pi A}{T}$$

$$A\sqrt{\frac{k}{m}} = \frac{2\pi A}{T}$$

$$\sqrt{\frac{k}{m}} = \frac{2\pi}{T}$$

$$\boldsymbol{T = 2\pi\sqrt{\frac{m}{k}}} \quad \text{or (since } f = \tfrac{1}{T}\text{)} \quad \boldsymbol{f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

<u>Note that the period and frequency of a mass-spring system DO NOT depend on the amplitude of its motion.</u>

Example 2: A 3.0 kg mass is attached to a spring of spring constant 80 N/m and is free to move along a horizontal, frictionless surface. The spring is initially stretched by a 10N force and released.

a)What is the amplitude of its motion?

b) What is its period?

c) What is its frequency?

d) How long will it take to reach the equilibrium point for the first time?

e) What is the maximum velocity of the mass?

f) What is the maximum acceleration of the mass?

Answers

a) Since F = kx, The 10N force will stretch the spring by an amount given by $x = \frac{F}{k} = \frac{10N}{80\frac{N}{m}}$ = 0.125m

b) $T = 2\pi\sqrt{\frac{m}{k}}$ $= 2\pi\sqrt{\frac{3kg}{80\frac{N}{m}}}$ = $2\pi\sqrt{.0375s^2}$ = 2π(0.19s) = 1.2s

c) f= $\frac{1}{T}$ = $\frac{1}{1.2s}$ = 0.82Hz

d) Traveling to the equilibrium point is ¼ of a cycle, so it will take 1.2s/4 = 0.30s

e) v_{max} = 2πAf = 2π(0.125m)(0.82Hz) = 0.64 m/s

f) $a_{max} = 4\pi^2Af^2 = 4\pi^2(0.125m)(0.82Hz)^2 = (39.4)(0.125m)(0.67Hz^2) = 3.3\ m/s^2$

The Relationship Between Velocity and Position

We can also use our energy analysis to determine the velocity of the mass at any location in its motion. We do this by combining two of our results:

$$U_{total} = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \qquad \text{and} \qquad U_{total} = \frac{1}{2}kA^2$$

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2$$

$$kx^2 + mv^2 = kA^2$$

$$mv^2 = kA^2 - kx^2$$

$$v^2 = \frac{k}{m}(A^2 - x^2)$$

$$v^2 = \sqrt{\frac{k}{m}(A^2 - x^2)}$$

Since $v_{max} = A\sqrt{\frac{k}{m}}$ and therefore $\frac{V_{max}}{A} = \sqrt{\frac{k}{m}}$ this can also be written as:

$$\boldsymbol{v = \frac{V_{max}}{A}\sqrt{A^2 - x^2}}$$

This provides the magnitude of the velocity of the mass at any location between +A and –A. Note that when x = A (when the mass is at its maximum distance from equilibrium) then v = 0. Also, when x = 0 (when the mass is passes through the equilibrium) that $v = v_{max}$.

At all other locations, the velocity is between $-v_{max}$ and $+v_{max}$.

Example 3: What is the magnitude of the velocity of the mass in Example 2 when it is located at x=0.062m?

Answer: $v = \frac{V_{max}}{A}\sqrt{A^2 - x^2}$

$$v = \frac{.64\frac{m}{s}}{.125m}\sqrt{(.125m)^2 - (.062m)^2}$$

$$v = 5.12s^{-1}\sqrt{.0156m^2 - .00384m^2}$$

$$v = 5.12s^{-1}\sqrt{.0118m^2}$$

$$v = (5.12\ s^{-1})(0.11m)$$

$$v = 0.56\ m/s$$

Extending these results to all SHM

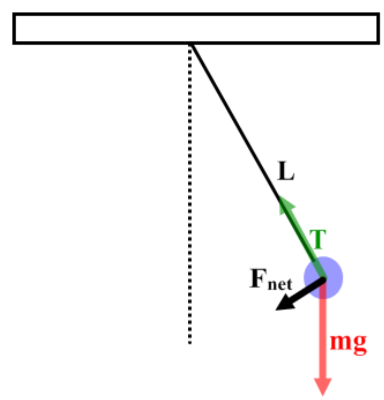
While our analysis was done for a mass-spring system it turns out that our results can be extended to all systems that undergo simple harmonic motion.

All objects that undergo SHM are subject to a force that takes the form F = -kx. That means that they all experience a "restoring force" directed back towards the equilibrium position that becomes stronger as the distance from equilibrium becomes greater.

Once it can be shown that the force acting on an object takes that form, then all the equations shown above apply. All that is necessary to apply them is to determine what plays the role of "k" and substitute that into the above formulas. A new set of relationships can then be derived for that particular case.

To some extent all restoring forces can be approximated as F=-kx for small displacement. As a result, SHM is found to be characteristic of many systems.

A good example of how to apply this approach is the pendulum.



The Pendulum

A pendulum is simply a mass at the end of a string or rod such that it is free to move from side to side. At equilibrium (when the mass is directly below its pivot point) the mass is pulled straight down by the force of gravity, mg. However, that force is completely offset by the upwards tension in the string. There is no net force acting on the object.

When the mass is pulled to the side, the force of gravity still pulls the mass straight down. However, that force cannot be totally offset by the tension in the string since the tension is now at an angle to the force of gravity; the result is a net force towards the equilibrium point, a restoring force.

With trigonometry it can be shown that for small angles that the amount of force is approximately given by:

$F = -mg\sin\theta$

$F = -mg(\frac{x}{L})$

$F = -(\frac{mg}{L})x$ (where L is the length of the pendulum)

The negative sign in this equation tells us that the force is a restoring force, that it will always pull the pendulum back towards its equilibrium point. That combined with the fact that the force is proportional to x, tells us that this will result in simple harmonic motion. We then have to determine what serves the role of "k", then we can use all our prior results.

F = -kx and $F = -(\frac{mg}{L})$x

$$-kx = -(\frac{mg}{L})x$$

$$k = \frac{mg}{L}$$

Once we have the "k" for this system, we can plug it into all of our previous equations to get the results for a pendulum.

$$T = 2\pi\sqrt{\frac{m}{k}}$$

[Substitute k = mg/L]

$$T = 2\pi\sqrt{\frac{m}{\frac{mg}{L}}}$$

$$\boldsymbol{T = 2\pi\sqrt{\frac{L}{g}}}$$

$$\boldsymbol{f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}}$$

Note that the period and frequency of a pendulum are independent of both its mass and its displacement, how far it travels in each cycle. It only depends on the local gravity, g, and the length of the string. This is why these became the foundation of early clocks and still be seen in "grandfather clocks."

The trigonometric relationships used to do the first step of this problem will be developed in your next physics course. But it is important to note that it needed to be assumed that the pendulum is not moving at a very large angle from the vertical; this is called a small angle approximation. A pendulum will only move in SHM for small displacements from the vertical.

Example 4: A pendulum consists of a 1.5 kg mass attached to the end of a 2.4m long string. The mass is pulled to the side and released.
a) What is its period?
b) What is its frequency?
c) How long will it take to swing to the same height on the opposite side?
d) If the mass were doubled how would these answers change?

Answers
a) $\boldsymbol{T = 2\pi\sqrt{\frac{L}{g}}}$= $\boldsymbol{T = 2\pi\sqrt{\frac{2.4m}{9.8\frac{m}{s^2}}}}$ = $\sqrt{.25s^2}$= 2π(0.50s) = 3.1s

b) f = $\frac{1}{T}$ = $\frac{1}{3.1s}$ = 0.32Hz

c) Traveling to the other maximum height is 1/2 of a cycle, so it will take $\frac{3.1s}{2}$ = 1.5s

d) No change; the mass is not a factor in the motion of a pendulum

Energy and the Pendulum

In all cases of SHM there are two forms of energy and a means for the efficient of conversion between them. In the case of the mass-spring system, it was kinetic energy (KE) and spring energy (U_{spring}). In the case of the pendulum it is **Gravitational Potential Energy (GPE)** and **Kinetic Energy (KE)**. The total energy of the system is constant, once it is set in motion, but the form of the energy varies.

$$U_{total} = \text{Constant}$$

$$U_{total} = \text{GPE} + \text{KE}$$

$$U_{total} = mgh = \frac{1}{2}mv^2$$

When a pendulum is at its lowest point (when the string is vertical) the GPE is defined to be zero, all the energy in the system is KE. When it reaches its highest point, its greatest displacement, the mass comes momentarily to rest and its KE is zero; all its energy is in the form of GPE. In between those two locations, the velocity can be any value between $-v_{max}$ and $+v_{max}$. It's total energy is given by either its maximum height (hmax) and by its maximum velocity (v_{max}).

$$U_{total} = mgh_{max} = \frac{1}{2}mv^2{}_{max}$$

$$mgh_{max} = \frac{1}{2}mv^2{}_{max}$$

$$gh_{max} = \frac{1}{2}v^2{}_{max}$$

$$\boldsymbol{v_{max} = \sqrt{2gh_{max}}} \qquad \text{OR} \qquad h_{max} = \frac{1}{2g}v^2{}_{max}$$

These equations allow you to find the maximum velocity or height of a pendulum bob given the other.

Example 5: What would be the magnitude of the velocity of the mass in Example 4 when it is located at its lowest point if it were initially pulled to the side such that it was elevated by 0.20m?

Answer: $v_{max} = \sqrt{2gh_{max}}$

$$v_{max} = \sqrt{2\left(9.8\frac{m}{s^2}\right)(.2m)}$$

$$v_{max} = \sqrt{3.92\frac{m^2}{s^2}}$$

v_{max} = 2.0 m/s

